Modeling Friction and Contact in Chrono

Theoretical Background
Things Covered

• Friction and contact, understanding the problem at hand

• The penalty approach

• The complementarity approach
Mass $\times$ Acceleration = Force
Mass $\times$ Acceleration = Force

- Coulomb friction coefficient - $\mu$

\[
m\ddot{v} = W + F + F_f + N
\]

$\|F_f\| \leq \mu\|N\|$ Reflect on this: friction force can assume a bunch of values (as long as they’re smaller than $\mu \times N$ though)
Additive Manufacturing (3D SLS Printing)

Courtesy of Professor Tim Osswald, Polymer Engineering Center, UW-Madison
Two main approaches: penalty & complementarity

Computational many-body dynamics
Handling frictional contact

Modelling approach
Penalty-based approach
Complementarity approach

Numerical techniques
Collision detection
Optimization techniques
General Comments, Penalty Approach

• Approach commonly used in handling granular material
  • Called “Discrete Element Method”

• The “Penalty” approach works well for sphere-to-sphere and sphere-to-plane scenarios
  • Deformable body mechanics used to characterize what happens under these scenarios

• Methodology subsequently grafted to general dynamics problem of rigid bodies – arbitrary geometry
  • When they collide, a fictitious spring-damper element is placed between the two bodies
    • Sometimes spring & damping coefficient based on continuum theory mentioned above
    • Sometimes values are guessed (calibration) based on experimental data
The Penalty Method, Taxonomy

• Depending on the normal relative velocity between bodies that experience a collision and their material properties, if there is no relative angular velocity, the collision is
  • Elastic, if the contact induced deformation is reversible and independent of displacement rate
  • Viscoelastic, if the contact induced deformation is irreversible, but the deformation is dependent on the displacement rate
  • Plastic, if collision leaves an involved body permanently deformed but the deformation of body is independent of the displacement rate
  • Viscoplastic, if impact is irreversible and similar to the viscoelastic contact but deformation depends on the displacement rate

• According to the dependency of the normal force on the overlap and the displacement rate, the force schemes can be subdivided into
  • Continuous potential models (like Lennard-Jones, for instance)
  • Linear viscoelastic models (simple, used extensively, what we use here)
  • Non-linear viscoelastic models
  • Hysteretic models (see papers of L. Vu-Quoc, in “DEM Further Reading” slide)
The Penalty Method in Chrono, Nuts and Bolts

- Method relies on a record (history) of tangential displacement $\delta_t$ to model static friction (see figure at right)
The Penalty Method in Chrono, Nuts and Bolts

\[ F_n = f \left( \frac{\delta_n}{D_{\text{eff}}} \right) \left( k_n \delta_n \mathbf{n} - \gamma_n m_{\text{eff}} \mathbf{v}_n \right) \]

\[ F_t = f \left( \frac{\delta_n}{D_{\text{eff}}} \right) \left( -k_t \delta_t - \gamma_t m_{\text{eff}} \mathbf{v}_t \right) \]

If \( |F_t| > \mu |F_n| \) then scale \(|\delta_t|\) so that \( |F_t| = \mu |F_n| \)

Visualize this \( \delta_t \) as creep.
Direct Shear Analysis via Granular Dynamics
[using LAMMPS/LIGGGHTS and Chrono]

- 1800 uniform spheres randomly packed
- Particle Diameter: $D = 5$ mm
- Shear Speed: 1 mm/s
- Inter-Particle Coulomb Friction Coefficient: $\mu = 0.5$ (Quartz on Quartz)
- Void Ratio (dense packing): $e = 0.4$
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DEM contact model in Chrono Parallel

Shear Stress / Normal Stress

Shear Displacement / Sphere Diameter

Chrono Serial, no history
Chrono Parallel, no history

[J. Fleischmann]→
Wave propagation in ordered granular material
Penalty Method – the Pros

• Backed by large body of literature and numerous validation studies

• No increase in the size of the problem
  • This is unlike the “complementarity” approach, discussed next

• Can accommodate shock wave propagation
  • Can’t do w/ “complementarity” approach since it’s a pure “rigid body” solution

• Easy to implement
  • Entire numerical solution decoupled
    • Easy to scale up to large problems
    • Parallel-computing friendly – run in parallel on per contact basis
      • Memory communication intensive
Penalty Method – Cons

1. Numerical stability requires small integration time steps
   • Long simulation times

2. Choice of integration time step strongly influences results

3. Sensitive wrt information provided by the collision detection engine

4. There is some hand-waving when it comes to arbitrary shapes and the fact that the friction force is a multi-valued function
DEM, Further Reading


The “Complementarity” Approach aka Differential Variational Inequality (DVI) Method
Two Shapes, and the Distance [Gap Function]

- Notation: $\partial A$ represents set of points making up the boundary of body $A$
- Shape body $A$: collection of points $S$ with $r_A^o = r_A + A_A \bar{s}_A^o$, $\bar{s}_A^o \in \partial A$
- Shape body $B$: collection of points $S$ with $r_B^o = r_B + A_B \bar{s}_B^o$, $\bar{s}_B^o \in \partial B$

- Signed distance function in a given configuration $q_A$ and $q_B$
  \[ \Phi(q_A(t), q_B(t)) \equiv \min_{\bar{s}_A^o \in \partial A, \bar{s}_B^o \in \partial B} ||r_A^o - r_B^o||_2 \]

- Contact when distance function is zero
  \[ \Phi(q_A(t^*), q_B(t^*)) = 0 \]
Body A – Body B Contact Scenario
Defining the Normal and Tangential Forces

- When a contact occurs: point of contact and local reference frame identified. Latter defined as follows:
  - $\mathbf{u}_i$ and $\mathbf{w}_i$ are two mutually perpendicular unit vectors in the tangent plan at the contact point
  - Unit vector $\mathbf{n}_i$ defines the normal direction in the local reference frame
- A normal force appears along the direction normal to the plane of contact
  - Magnitude of the force is $\hat{\gamma}_{i,n}$. Specifically,
    \[ \mathbf{F}_{i,N} = \hat{\gamma}_{i,n} \mathbf{n}_i \]
- A friction force appears in the tangent plane
  - Has two components along the axes $\mathbf{u}_i$ and $\mathbf{w}_i$: $\hat{\gamma}_{i,u}$ and $\hat{\gamma}_{i,w}$, respectively. Specifically,
    \[ \mathbf{F}_{i,T} = \hat{\gamma}_{i,u} \mathbf{u}_i + \hat{\gamma}_{i,w} \mathbf{w}_i \]
- NOTE: The point of contact, $\mathbf{n}_i$, $\mathbf{u}_i$, and $\mathbf{w}_i$ are obtained at the end of the collision detection task, which is executed at the beginning of each time step
DVI-Based Methods: The **Contact** Model

- A contact is modeled by one inequality constraint, which states that either the distance between two bodies is greater than zero $\Phi_i(q) > 0$, in which case the normal force is zero $\tilde{\gamma}_{i,n} = 0$, or vice-versa; i.e., if the distance is zero, the contact force is nonzero.

  - Condition above captured in the following complementarity condition:

    $$\tilde{\gamma}_{i,n} \geq 0, \quad \Phi_i(q) \geq 0, \quad \Phi_i(q)\tilde{\gamma}_{i,n} = 0,$$

  - Another way to state the complementarity condition:

    $$0 \leq \tilde{\gamma}_{i,n} \perp \Phi_i(q) \geq 0$$
DVI-Based Methods: The Friction Model

- The friction model considered is Coulomb’s:

\[ \mu_i \tilde{\gamma}_{i,n} \geq \sqrt{\tilde{\gamma}_{i,u}^2 + \tilde{\gamma}_{i,w}^2} \]

\[ \textbf{F}_{i,T}^T \cdot \textbf{v}_{i,T} = - \| \textbf{F}_{i,T} \| \| \textbf{v}_{i,T} \| \]

\[ \| \textbf{v}_{i,T} \| \left( \mu_i \tilde{\gamma}_{i,n} - \sqrt{\tilde{\gamma}_{i,u}^2 + \tilde{\gamma}_{i,w}^2} \right) = 0 \]

- First condition: friction force is within the friction cone

- Second condition: friction force and tangential velocity between two bodies at point of contact are collinear and of opposite direction

- The third condition captures the stick-slip condition. If the velocity is greater than zero, it means that the friction force saturated; i.e., \( \mu_i \tilde{\gamma}_{i,n} = \sqrt{\tilde{\gamma}_{i,u}^2 + \tilde{\gamma}_{i,w}^2} \); this is the sliding scenario. Conversely, if the bodies stick to each other, then the relative tangential velocity is zero, \( \textbf{v}_{i,T} = \textbf{0} \), and the friction force is not saturated \( \mu_i \tilde{\gamma}_{i,n} > \sqrt{\tilde{\gamma}_{i,u}^2 + \tilde{\gamma}_{i,w}^2} \).
Coulomb’s Model Posed as the Solution of an Optimization Problem

- Assume that \( \tilde{\gamma}_{i,n} \) and \( v_{i,T} \) are given and you pose the following optimization problem in variables \( x \) and \( y \):
  - Minimize the function \( v_{i,T}^T (xu_i + yw_i) \) subject to the constraint \( \sqrt{x^2 + y^2} \leq \mu_i \tilde{\gamma}_{i,n} \)

- If you pose the first order Karush-Kuhn-Tucker optimality conditions for this optimization problem you end up precisely with the set of three conditions that define the Coulomb friction model

- It follows that there is an interplay between \( \tilde{\gamma}_{i,n} \), \( \tilde{\gamma}_{i,u} \), \( \tilde{\gamma}_{i,w} \), and \( v_{i,T} \). Using math notation

\[
(\tilde{\gamma}_{i,u}, \tilde{\gamma}_{i,w}) = \arg\min_{\sqrt{x^2 + y^2} \leq \mu_i \tilde{\gamma}_{i,n}} v_{i,T}^T (xu_i + yw_i).
\]
The DVI Problem: The EOM, in Fine-Granularity Form

- Time evolution of the dynamical system is the solution of the following DVI problem:

\[
B = 1, \ldots, nb : \quad m_B \ddot{\mathbf{r}}_B = \sum_{i \in B(B)} \left[ \Psi_{xB}^{(i)} \right]^T \hat{\gamma}_{i,b} + f_B(t, \mathbf{q}, \mathbf{v}) + \sum_{i \in A(B)} \left( \hat{\gamma}_{i,n} \mathbf{n}_i + \hat{\gamma}_{i,u} \mathbf{u}_i + \hat{\gamma}_{i,w} \mathbf{w}_i \right)
\]

\[
B = 1, \ldots, nb : \quad \mathbf{J}_B \dot{\omega}_B = \sum_{i \in B(B)} \hat{\mathbf{P}}_B^T(\Psi^{(i)}) \hat{\gamma}_{i,b} + \tau_B(t, \mathbf{q}, \mathbf{v}) + \sum_{i \in A(B)} \hat{s}_{i,B} \mathbf{A}_B^T \left( \hat{\gamma}_{i,n} \mathbf{n}_i + \hat{\gamma}_{i,u} \mathbf{u}_i + \hat{\gamma}_{i,w} \mathbf{w}_i \right)
\]

\[
B = 1, \ldots, nb : \quad \dot{\mathbf{p}}_B = \frac{1}{2} \mathbf{G}^T(\mathbf{p}_B) \dot{\omega}_B
\]

\[
i \in B : \quad \Psi_i(\mathbf{q}, t) = 0
\]

\[
i \in A : \quad 0 \leq \hat{\gamma}_{i,n} \perp \Phi_i(\mathbf{q}) \geq 0,
\]

\[
i \in A : \quad (\hat{\gamma}_{i,u}, \hat{\gamma}_{i,w}) = \arg\min_{\sqrt{x^2+y^2} \leq \mu_i \hat{\gamma}_{i,n}} v^T (x D_{i,u} + y D_{i,w})
\]
Frictional Contact: The Matrix-Vector Form

- Problem on previous slide reformulated using matrix-vector notation, assumes form

\begin{align*}
\dot{q} &= L(q)v \\
M\dot{v} &= f(t, q, v) + \sum_{i \in B} \tilde{\tau}_{i,b} \nabla \Psi_i + \sum_{i \in A} (\tilde{\tau}_{i,n} D_{i,n} + \tilde{\tau}_{i,u} D_{i,u} + \tilde{\tau}_{i,w} D_{i,w}) \\
& \quad \text{for } i \in B: \Psi_i(q, t) = 0 \\
& \quad \text{for } i \in A: 0 \leq \tilde{\tau}_{i,n} \perp \Phi_i(q) \geq 0, \\
(\tilde{\tau}_{i,u}, \tilde{\tau}_{i,w}) &= \arg\min_{\sqrt{x^2+y^2} \leq \mu_i \tilde{\tau}_{i,n}} v^T (x D_{i,u} + y D_{i,w})
\end{align*}
The Discretization Process

- For straight index-3 DAE solution (like ADAMS), one uses the Newton-Euler form of the equations of motion in conjunction with the level zero constraints (the position constraint equations).

- The DVI solution relies on the level one constraints (velocity level constraints).

- Implications:
  
  - Since the level zero constraints are not enforced, there will be drift in the solution.
  - Stabilization terms, that penalize the violation of the level zero constraints, are added to the level one bilateral and unilateral constraints.
  - Bilateral and unilateral constraints massaged into the following (superscript $(l)$ denotes the time step):

    \[
    \begin{align*}
    i \in B & : \quad \frac{1}{h} \Psi_i(q^{(l)}, t) + \nabla \Psi_i^T v^{(l+1)} + \frac{\partial \Psi_i}{\partial \dot{q}} = 0 \\
    i \in A & : \quad 0 \leq \gamma_{i,n} \perp \frac{1}{h} \Phi_i(q^{(l)}) + D_{i,n}^T v^{(l+1)} \geq 0.
    \end{align*}
    \]

    * Reminiscent of a Baumgarte stabilization scheme.
The Discretization Process

- The discretized form of the DVI problem:

\[ M(v^{(t+1)} - v^{(t)}) = hf(t^{(t)}, q^{(t)}, v^{(t)}) + \sum_{i \in B} \gamma_i \nabla \varphi_i + \sum_{i \in A} (\gamma_i n D_{i,n} + \gamma_i w D_{i,w}) \]

\[ i \in B : \quad \frac{1}{h} \varphi_i (q^{(t)}, t) + \nabla \varphi_i T v^{(t+1)} + \frac{\partial \varphi_i}{\partial t} = 0 \]

\[ i \in A : \quad 0 \leq \gamma_i n \perp \frac{1}{h} \Phi_i (q^{(t)}) + D^T_{i,n} v^{(t+1)} = 0 \]

\[ \begin{align*}
(\gamma_{i,n}, \gamma_{i,w}) &= \arg\min_{\mu \gamma_{i,n} \geq \sqrt{x^2 + y^2}} v^T (x D_{i,n} + y D_{i,w}) \\
q^{(t+1)} &= q^{(t)} + hL(q^{(t)})v^{(t+1)}.
\end{align*} \]

- The first four of the equations above together combine for an optimization problem with equilibrium constraints

- Why an optimization problem?
  - Because the way the Coulomb friction model is posed

- What type of optimization problem?
  - This represents a nonlinear optimization problem
  - Can be linearized if the friction cone is discretized and represented as a multifaceted pyramid (problem size increases & anisotropy creeps in)

- What are the 'equilibrium constraints'?
  - Your typical optimization problem might display algebraic equality or inequality constraints
  - Above, we are solving an optimization problem for which the constraints represent the discretization of a set of differential equations
The NCP → CCP Metamorphosis

- Dealing with some generic nonlinear optimization problem like the one above is daunting

- Trick used to recast it as a simpler optimization problem for which
  (i) We are guaranteed that a solution exists (ideally, it would be unique, in some sense), and
  (ii) There are tailored algorithms that we can use to efficiently find the solution

- Trick (coming from the left field): introduce a relaxation of the complementarity constraints
  
  Instead of working with this:
  \[ i \in A : 0 \leq \gamma_i \quad \text{and} \quad \frac{1}{h} \Phi_i(q^{(t)}) + D_{i,u}^T v^{(t+1)} \geq 0 \]

  Work with this:
  \[ i \in A : 0 \leq \gamma_i \quad \text{and} \quad \frac{1}{h} \Phi_i(q^{(t)}) + D_{i,u}^T v^{(t+1)} - \mu_i \sqrt{(v^T D_{i,u})^2 + (v^T D_{i,w})^2} \geq 0 \]

- Owing to this relaxation, the NCP problem becomes a cone complementarity problem (CCP)
The Cone Complementarity Problem

- The relaxed problem we have to deal with now looks like this

\[
M(v^{(l+1)} - v^{(l)}) = hf(t^{(l)}, q^{(l)}, v^{(l)}) + \sum_{i \in B} \gamma_{i,b} \nabla \Psi_i + \sum_{i \in A} (\gamma_{i,n} D_{i,n} + \gamma_{i,u} D_{i,u} + \gamma_{i,w} D_{i,w})
\]

\[
i \in B : \quad \frac{1}{h} \Psi_i(q^{(l)}, t) + \nabla \Psi_i^T v^{(l+1)} + \frac{\partial \Psi_i}{\partial t} = 0
\]

\[
i \in A : \quad 0 \leq \gamma_{i,u} \perp \frac{1}{h} \Phi_i(q^{(l)}) + D_{i,n}^T v^{(l+1)} - \mu_i \sqrt{(v^T D_{i,u})^2 + (v^T D_{i,w})^2} \geq 0
\]

\[
(\gamma_{i,u}, \gamma_{i,w}) = \arg\min_{\sqrt{x^2+y^2} \leq \mu_i \gamma_{i,n}} v^T (x D_{i,u} + y D_{i,w})
\]

\[
q^{(l+1)} = q^{(l)} + hL(q^{(l)}) v^{(l+1)}.
\]
Cone Complementarity Problem (CCP)

- After some algebraic massaging, the equations on the previous slide combine to lead to the following CCP:

  - Introduce the convex hypercone...

    \[ \Upsilon = \left( \bigoplus_{i \in \mathcal{A}(q(t))} \mathcal{F}C^i \right) \oplus \left( \bigoplus_{i \in \mathcal{B}(q(t))} \mathcal{B}C^i \right) \]

    where \( \mathcal{F}C^i \) is the \( i \)-th friction cone\( \mathcal{B}C^i \) is \( \mathbb{R} \)

  - ... and its polar hypercone

    \[ \Upsilon^\circ = \left( \bigoplus_{i \in \mathcal{A}(q(t))} \mathcal{F}C_i^\circ \right) \oplus \left( \bigoplus_{i \in \mathcal{B}(q(t))} \mathcal{B}C_i^\circ \right) \]

  - The CCP that needs to be solved at each time step is as follows:

    * Find the Lagrange hyper-multiplier \( \gamma \) that satisfies:

      \[ \Upsilon \ni \gamma \perp -(N\gamma + r) \in \Upsilon^\circ \]

    * The matrix \( N \) and vector \( r \) are given, computed based on state information at time-step \( t(q) \)
The Optimization Angle

- CCP represents first order optimality condition (KKT conditions) for a quadratic problem with conic constraints

\[
\min_{\gamma} \frac{1}{2}\gamma^T N \gamma + r^T \gamma
\]

subject to \( \gamma_i \in \mathcal{C}_i \) for \( i = 1, 2, \ldots, N_c \).

- \( N \in \mathbb{R}^{3N_c \times 3N_c} \) is symmetric and positive semi-definite
- \( N \) and \( r \in \mathbb{R}^{3N_c} \) do not depend on \( \gamma \). They are computed once at the beginning of each time step
- The problem is convex, therefore it has a global solution
- Problem does not have a unique solution (since \( N \) is not positive-definite)
Wrapping it Up, Complementarity Approach

- Everything straightforward once frictional contact forces are available
  
  - The velocity $v^{(l+1)}$ is computed via a matrix-vector multiplication
  
  - Once velocity available, generalized positions $q^{(l+1)}$ computed as
    \[ q^{(l+1)} = q^{(l)} + hL(q^{(l)})v^{(l+1)} \]
Complementarity Approach: Putting Things in Perspective

• Perform collision detection

• Formulate equations of motion; i.e., pose DVI problem

• DVI discretized to lead to nonlinear complementarity problem (NCP)

• Relax NCP to get CCP

• Equivalently, solve QP with conic constraints to compute $\gamma$

• Once friction and contact forces available, velocity available

• Once velocity available, positions are available (numerical integration)
Additive Manufacturing (3D SLS Printing)

 Courtesy of Professor Tim Osswald, Polymer Engineering Center, UW-Madison
Selective Laser Sintering (SLS) Layering

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<th>Granular Material</th>
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<tr>
<td>$N$</td>
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<td>$\rho$</td>
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<td>$r_{\text{mean}}$</td>
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<tr>
<td>$r_{\sigma}$</td>
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<table>
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<tr>
<td>$\Delta t$</td>
<td>$5 \times 10^{-5}$ [s]</td>
</tr>
<tr>
<td>Run Time</td>
<td>49 Hours</td>
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</tbody>
</table>
Dress 3D Printing Problem
Using Simulation in 3D Printing of Clothes
Pros and Cons, Complementarity Approach

• Pros
  • Allows for large integration step sizes since it doesn’t have to deal with contact stiffness
  • Reduced number of model parameters one can fiddle with
  • It looks at the entire problem, it doesn’t artificially decouples the problem

• Cons
  • Requires a global solution, which means that large systems lead to large coupled problems
  • Our implementation has numerical artifacts owing to the relaxation of the non-penetration condition
  • Challenging to model coefficient of restitution (currently uses an inelastic model)
  • Stuck w/ a rigid body dynamics take on the problem (can’t propagate shock waves)
Reference, DVI Literature

• Lab technical report:


Closing Remarks  
[Applies both for Penalty and DVI approaches]

- There is some hand waving when it comes to handling friction and contact  
  - Both in Penalty and DVI

- Handling frictional contact is equally art and science  
  - To get something to run robustly requires tweaking  
  - Takes some time to understand strong/weak points of each approach

- Continues to be area of active research
Supplemental Slides
Notation Conventions

[1/2]

- To stick with the presentation in the paper of Anitescu and Tasora, we’ll use the following notation

  - Point 1: Instead of sticking with transpose of Jacobians, we’ll use gradients, which are defined precisely as the transpose of the Jacobians. Specifically,

    \[ \nabla_q \Psi_i = \Psi_{i,q}^T = [\partial \Psi_i / \partial q]^T \quad \text{and} \quad \nabla_q \Phi_i = \Phi_{i,q}^T = [\partial \Phi_i / \partial q]^T \nabla \]

  - Point 2: We’ll use the transformation matrix \( L(q) \) to link the time derivative of the level zero unknowns in the \( r - p \) formulation to the level one unknowns in the \( r - \omega \) formulation:

    \[ \dot{q} = L(q)v \]

  - Point 3: To keep the notation simpler (and probably confuse you), we’ll group the translational and rotational equations of motion in one big matrix-vector equation (nothing changed, except the notation) in order to have less symbols and equations to deal with

  - Point 4: We’ll use the following notation (\( h \) is the integration step-size)

    \[ \gamma_{i,n} = h\tilde{\gamma}_{i,n} \quad \gamma_{i,u} = h\tilde{\gamma}_{i,u} \quad \gamma_{i,w} = h\tilde{\gamma}_{i,w} \quad \gamma_{i,b} = h\tilde{\gamma}_{i,b} \]

    * Recall that time \( \times \) force (like in \( \gamma_{i,n} = h\tilde{\gamma}_{i,n} \)) is impulse, and it’s impulse that changes the momentum of a body
• Define the transformation matrix $A_i$ that given the representation of a geometric vector in the contact reference frame associated with contact $i$ is used to generate its representation in the GRF:

$$A_{i\rightarrow G} = \begin{bmatrix} n_i & u_i & w_i \end{bmatrix}$$

Note that the frictional contact force at contact $i$ as felt by body $A$ is simply

$$F_{i,A}^c = n_i \hat{\gamma}_{i,n} + u_i \hat{\gamma}_{i,u} + w_i \hat{\gamma}_{i,w} = \begin{bmatrix} n_i & u_i & w_i \end{bmatrix} \begin{bmatrix} \hat{\gamma}_{i,n} \\ \hat{\gamma}_{i,u} \\ \hat{\gamma}_{i,w} \end{bmatrix} = A_{i\rightarrow G} \cdot \hat{\gamma}_i$$

where $\hat{\gamma}_i = \begin{bmatrix} \hat{\gamma}_{i,n} \\ \hat{\gamma}_{i,u} \\ \hat{\gamma}_{i,w} \end{bmatrix}$

• A projection matrix $D_i$ is defined for each contact $i \in A$ to project the contact forces onto the equations of motion, both for translation and rotation. If we assume that contact $i$ acts between body $A$ and body $B$,

$$D_i \equiv \begin{bmatrix} 0 \\ \vdots \\ A_{i\rightarrow G} \\ \tilde{s}_{i,A} A_A^T A_{i\rightarrow G} \\ 0 \\ \vdots \\ 0 \\ -A_{i\rightarrow G} \\ -\tilde{s}_{i,B} A_B^T A_{i\rightarrow G} \\ 0 \end{bmatrix}_{6nb \times 3}$$

Notation used in expression of $D_i$: the vectors $\tilde{s}_{i,A}$ and $\tilde{s}_{i,B}$ represent the location of the contact point in the local reference frame of body $A$ and $B$, respectively.

• The columns of $D_i$ are denoted by $D_{i,n}$, $D_{i,u}$, $D_{i,w}$ and are each vectors of dimension $6nb$:

$$D_i = \begin{bmatrix} D_{i,n} & D_{i,u} & D_{i,w} \end{bmatrix}_{6nb \times 3}$$
General Comments, DVI

• Differential Variational Inequality (DVI): a set of differential equations that hold in conjunction with a collection of constraints

  • Classical equations of motion: Newton-Euler EOMs, govern time evolutions of constrained MBS

  • Kinematic constraints coming from joints
    • These constraints are called bilateral constraints

  • When dealing with contacts, the non-penetration condition captured as a unilateral constraint
    • At point of contact, relative to body 1, body 2 can move outwards, but not inwards

  • The variational attribute stems from the optimization problem posing the Coulomb friction model
Bilateral vs. Unilateral Constraints

- Nomenclature: classical MBD uses kinematic constraints, which we’ll call bilateral constraints. In DVI we also have non-penetration constraints, which are unilateral constraints and assume the form of inequalities.

- Notation: We’ll call $\mathcal{A}$ the set of all active unilateral constraints present in the system. Think of these as active contacts. They’ll be denoted by

  \[
  \Phi_i(q) \quad i \in \mathcal{A}
  \]

  Note that the nonpenetration condition is expressed as (the distance between two bodies should also be positive)

  \[
  \Phi_i(q) \geq 0, \quad i \in \mathcal{A}
  \]

- Notation: We’ll call $\mathcal{B}$ the set of all bilateral constraints present in the system. These expression of these constraints will be denoted by $\Psi(q, t)$. Just like before we have that

  \[
  \Psi_i(q, t) = 0, \quad i \in \mathcal{B}
  \]

- Remark: While the bilateral constraints typically don’t change in time (a spherical joint stays a spherical joint throughout the simulation), the unilateral constraints appear and disappear; i.e., contacts are made and then broken. In other words, $\mathcal{A}$ depends on the state $q$ of the system.