General Theoretical Concepts Related to Multibody Dynamics
Before Getting Started

• Material draws on two main sources


  • Course notes, available at: http://sbel.wisc.edu/Courses/ME751/2016/
Looking Ahead

• Purpose of this segment:
  • Quick discussion of several theoretical concepts that come up time and again when using Chrono

• Concepts covered
  • Reference frames and changes of reference frames
  • Elements of the kinematics of a 3D body (position, velocity and acceleration of a body)
  • Kinematic constraints (joints)
  • Formulating the equations of motion
    • Newton-Euler equations of motion (via D’Alembert’s Principle)
Reference Frames in 3D Kinematics. Problem Setup

• Global Reference Frame (G-RF) attached to ground at point O

• Imagine point P is fixed (red-pen mark) on the rigid body

• Rigid body has a reference frame attached (fixed) to it
  • Assume its origin is at O (same as G-RF)
  • Called Local Reference Frame (L-RF) – shown in blue
  • Axes: $f, g, h$

• Question of interest:
  • What is the relationship between the coordinates of point P in G-RF and L-RF?
More Formal Way of Posing the Question

- Let \( \vec{q} = \overrightarrow{OP} \) be a geometric vector (see figure).

- In the G-RF defined by (\( \vec{i}, \vec{j}, \vec{k} \)), the geometric vector \( \vec{q} \) is represented as
  \[
  \vec{q} = q_x \vec{i} + q_y \vec{j} + q_z \vec{k}
  \]

- In the L-RF defined by (\( \vec{f}, \vec{g}, \vec{h} \)), the geometric vector \( \vec{q} \) is represented as
  \[
  \vec{q} = \bar{q}_x \vec{f} + \bar{q}_y \vec{g} + \bar{q}_z \vec{h}
  \]

- QUESTION: how are \((q_x, q_y, q_z)\) and \((\bar{q}_x, \bar{q}_y, \bar{q}_z)\) related?
Relationship Between L-RF Vectors and G-RF Vectors

\[ \vec{f}' = a_{11} \vec{i} + a_{21} \vec{j} + a_{31} \vec{k} \]
\[ \vec{g}' = a_{12} \vec{i} + a_{22} \vec{j} + a_{32} \vec{k} \]
\[ \vec{h}' = a_{13} \vec{i} + a_{23} \vec{j} + a_{33} \vec{k} \]

\[
\begin{bmatrix}
  a_{11} \\
  a_{21} \\
  a_{31}
\end{bmatrix}
\]
\[
\begin{bmatrix}
  a_{12} \\
  a_{22} \\
  a_{32}
\end{bmatrix}
\]
\[
\begin{bmatrix}
  a_{13} \\
  a_{23} \\
  a_{33}
\end{bmatrix}
\]

\[ a_{11} = \vec{i} \cdot \vec{f}' = \cos \theta(\vec{i}, \vec{f}') \]
\[ a_{12} = \vec{i} \cdot \vec{g}' = \cos \theta(\vec{i}, \vec{g}') \]
\[ a_{13} = \vec{i} \cdot \vec{h}' = \cos \theta(\vec{i}, \vec{h}') \]

\[ a_{21} = \vec{j} \cdot \vec{f}' = \cos \theta(\vec{j}, \vec{f}') \]
\[ a_{22} = \vec{j} \cdot \vec{g}' = \cos \theta(\vec{j}, \vec{g}') \]
\[ a_{23} = \vec{j} \cdot \vec{h}' = \cos \theta(\vec{j}, \vec{h}') \]

\[ a_{31} = \vec{k} \cdot \vec{f}' = \cos \theta(\vec{k}, \vec{f}') \]
\[ a_{32} = \vec{k} \cdot \vec{g}' = \cos \theta(\vec{k}, \vec{g}') \]
\[ a_{33} = \vec{k} \cdot \vec{h}' = \cos \theta(\vec{k}, \vec{h}') \]

There is a good reason the values \( a_{ij} \) above are called "direction cosines".
Punch Line, Change of Reference Frame (from “source” to “destination”)

\[
\begin{bmatrix}
q_x \\
q_y \\
q_z
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
\bar{q}_x \\
\bar{q}_y \\
\bar{q}_z
\end{bmatrix}
\]

\[q_d = A_{ds} q_s\]

\[A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix} = [f \quad g \quad h]\]

\[f = \begin{bmatrix}
a_{11} \\
a_{21} \\
a_{31}
\end{bmatrix} \quad g = \begin{bmatrix}
a_{12} \\
a_{22} \\
a_{32}
\end{bmatrix} \quad h = \begin{bmatrix}
a_{13} \\
a_{23} \\
a_{33}
\end{bmatrix}\]
The Bottom Line: Moving from RF to RF

• Representing the same geometric vector in two different RFs leads to the concept of “rotation matrix”, or “transformation matrix” $A_{ds}$:

  • Getting the new coordinates, that is, representation of the same geometric vector in the new RF is as simple as multiplying the coordinates by the rotation matrix $A_{ds}$:

  $$\mathbf{q} = A_{ds} \tilde{\mathbf{q}}$$

• NOTE 1: what is changed is the RF used to represent the vector
  • We are talking about the *same* geometric vector, represented in two RFs

• NOTE 2: rotation matrix $A_{ds}$ sometimes called “orientation matrix”
Rotation Matrix is Orthogonal

- Recall that $\vec{f}$, $\vec{g}$, and $\vec{h}$ are mutually orthogonal
- Recall that $\vec{f}$, $\vec{g}$, and $\vec{h}$ are unit vectors
- Therefore, the following holds:

\[
\begin{align*}
  f^T f &= g^T g = h^T h = 1 \\
  f^T g &= g^T h = h^T f = 0
\end{align*}
\]

- Consequently, the rotation matrix $A$ is orthogonal

\[
A^T A = AA^T = I_{3 \times 3}
\]
Summarizing Key Points, Reference Frames

- Started with the representation $\mathbf{q}_s$ of a geometric vector $\mathbf{q}$ in a “source” reference frame $s$

- The representation of the geometric vector $\mathbf{q}$ in a “destination” reference frame $d$ is given by

  $$\mathbf{q}_d = A_{ds} \mathbf{q}_s$$

- Matrix $A_{ds}$ called transformation, or rotation matrix (taking vector from the source RF $s$ to the destination RF $d$)

- Because $A_{ds}$ is orthogonal, one has that

  $$\mathbf{q}_s = A_{ds}^T \mathbf{q}_d$$  therefore  $$A_{sd} = A_{ds}^T$$

- Many times, the “destination” RF is the global reference frame (G-RF), which has ID “0”
  - In this case, we don’t show “0” anymore, simply call $A_s$ instead of $A_{0s}$
**New Topic:**
Angular Velocity. 3D Problem Setup

- Global Reference Frame (G-RF) attached to ground at point O
- Imagine point P is fixed (red-pen mark) on the rigid body
- Rigid body has a reference frame attached to it
  - Assume its origin is at O (same as G-RF)
  - Local Reference Frame (L-RF) – shown in blue
  - Axes: $\mathbf{f}, \mathbf{g}, \mathbf{h}$
- Question of interest:
  - How do we express rate of change of blue RF wrt global RF?
Angular Velocity, Getting There...

- Recall that $\mathbf{A}_i \mathbf{A}_i^T = \mathbf{I}_{3 \times 3}$. Taking a time derivative yields

$$\dot{\mathbf{A}}_i \mathbf{A}_i^T + \mathbf{A}_i \dot{\mathbf{A}}_i^T = \mathbf{0}_{3 \times 3} \quad \Rightarrow \quad \dot{\mathbf{A}}_i \mathbf{A}_i^T = -\mathbf{A}_i \dot{\mathbf{A}}_i^T$$

- Quick remarks
  - The matrix $\dot{\mathbf{A}}_i \mathbf{A}_i^T$ is a $3 \times 3$ matrix
  - The matrix $\dot{\mathbf{A}}_i \mathbf{A}_i^T$ is skew-symmetric

- CONCLUSION: there must be a vector, $\omega_i$, whose cross product matrix is equal to the $3 \times 3$ skew symmetric matrix $\dot{\mathbf{A}}_i \mathbf{A}_i^T$:

$$\tilde{\omega}_i = \dot{\mathbf{A}}_i \mathbf{A}_i^T$$

- This vector $\omega_i$ is called the angular velocity of the L-RF with respect to the G-RF.
Angular Velocity: Represented in G-RF or in L-RF

- Since $A_i$ is orthogonal, rate of change $\dot{A}_i$ of orientation matrix is simply
  \[
  \dot{A}_i = \tilde{\omega}_i A_i
  \]

- Angular velocity vector can be represented in the local reference frame. Skipping details,
  \[
  \tilde{\omega}_i = A_i^T \dot{A}_i
  \]

- Therefore, rate of change $\dot{A}_i$ of orientation matrix can also be represented as
  \[
  \dot{A}_i = A_i \tilde{\omega}_i
  \]

- Notation convention: an over-bar placed on a vector (like $\bar{\omega}_i$ above) indicates that quantity
  is a representation of a geometric vector in a local reference frame
New Topic:
Using Euler Parameters to Define Rotation Matrix A

• **Starting point: Euler’s Theorem**
  “If the origins of two right-hand Cartesian reference frames coincide, then the RFs may be brought into coincidence by a single rotation of a certain angle $\chi$ about a carefully chosen unit axis $\mathbf{u}$”

• **Euler’s Theorem proved in the following references:**
  • Wittenburg – Dynamics of Systems of Rigid Bodies (1977)
Warming up...

- Green color - used for quantities that define the Euler rotation: the axis of rotation defined by the unit vector \( \vec{u} \) and the angle \( \chi \)

- Red color - used to indicate the vectors that need to be summed up to get axis \( \vec{h} \) of the L-RF

- Blue color - denotes the \( \vec{f} - \vec{g} - \vec{h} \) axes of the L-RF

- Black dotted line - support entities (helpers, don’t play any role but only help with the derivation). The angle \( \alpha \) measured between the axis of rotation \( \vec{u} \) and the \( \vec{k} \) unit vector.

- **Other notation used:** \( ||\vec{a}|| = a \quad ||\vec{b}|| = b \quad ||\vec{c}|| = c \)
How Euler Parameters Come to Be

- Using as input $\chi$ and $\mathbf{u}$, one can express the vectors $\vec{f}$, $\vec{g}$, and $\vec{h}$ in the global reference frame as

\[
\begin{align*}
\vec{f} &= i(2\cos^2\frac{\chi}{2} - 1) + 2\mathbf{u}(\mathbf{u}^T i)\sin^2\frac{\chi}{2} + 2ui\sin\frac{\chi}{2}\cos\frac{\chi}{2} \\
\vec{g} &= j(2\cos^2\frac{\chi}{2} - 1) + 2\mathbf{u}(\mathbf{u}^T j)\sin^2\frac{\chi}{2} + 2uj\sin\frac{\chi}{2}\cos\frac{\chi}{2} \\
\vec{h} &= k(2\cos^2\frac{\chi}{2} - 1) + 2\mathbf{u}(\mathbf{u}^T k)\sin^2\frac{\chi}{2} + 2uk\sin\frac{\chi}{2}\cos\frac{\chi}{2}
\end{align*}
\]

- The expression of $\vec{f}$, $\vec{g}$, and $\vec{h}$ justifies the introduction of the following generalized coordinates (the “Euler Parameters”):

\[
\mathbf{p} = \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix}
\]

where $e_0 = \cos\frac{\chi}{2}$ and $\mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = u \sin\frac{\chi}{2}$

- Note: $\mathbf{u}$ unit vector $\Rightarrow$ values of $e_0$, $e_1$, $e_2$, and $e_3$ must satisfy the normalization condition

\[
e_0^2 + e_1^2 + e_2^2 + e_3^2 = e_0^2 + \mathbf{e}^T \mathbf{e} = 1
\]
Orientation Matrix, Based on Euler Parameters

- Based on definition of $e_0$, $e_1$, $e_2$, and $e_3$,

$$ f = [(2e_0^2 - 1)I + 2(ee^T + e_0\vec{e})]i $$

$$ g = [(2e_0^2 - 1)I + 2(ee^T + e_0\vec{e})]j $$

$$ h = [(2e_0^2 - 1)I + 2(ee^T + e_0\vec{e})]k $$

- Recall that $A = [f \ g \ h]$

- Therefore,

$$ A = [(2e_0^2 - 1)I + 2(ee^T + e_0\vec{e})] $$

- Equivalently,

$$ A = 2 \begin{bmatrix}
  e_0^2 + e_1^2 - \frac{1}{2} & e_1 e_2 - e_0 e_3 & e_1 e_3 + e_0 e_2 \\
  e_1 e_2 + e_0 e_3 & e_0^2 + e_2^2 - \frac{1}{2} & e_2 e_3 - e_0 e_1 \\
  e_1 e_3 - e_0 e_2 & e_2 e_3 + e_0 e_1 & e_0^2 + e_3^2 - \frac{1}{2}
\end{bmatrix} $$
New Topic:
Beyond Rotations – Full 3D Kinematics of Rigid Bodies

- So far, focus was only on the rotation of a rigid body

- Body connected to ground through a spherical joint
  - Body experienced an arbitrary rotation

- Yet bodies are experiencing both translation and rotation
3D Kinematics of Rigid Body: Problem Backdrop

- Framework and Notation Conventions:
  - A L-RF is attached to the rigid body at some location denoted by $O'$
  - Relative to the G-RF, point $O'$ is located by vector $\vec{r}$
  - L-RF defined by vectors $\vec{f}, \vec{g}, \vec{h}$
  - An arbitrary point $P$ of the rigid body is considered. Its location relative to the L-RF is provided through the vector $\vec{s}^P$
3D Rigid Body Kinematics: Position of an Arbitrary Point \( P \)

- **The Geometric View:**
  \[
  \overrightarrow{OP} = \overrightarrow{OO'} + \overrightarrow{O'P}
  \]
  \[
  \overrightarrow{r'}^P = \overrightarrow{r} + \overrightarrow{s}^P
  \]

- **The Algebraic Representation:**
  \[
  \overrightarrow{r'}^P = \overrightarrow{r} + \overrightarrow{s}^P = \overrightarrow{r} + A\overrightarrow{s}^P
  \]

- **Important observation:**
  - The vector \( \overrightarrow{s}^P \) that provides the location of \( P \) in the L-RF is a constant vector
    * True because the body is assumed to be rigid
3D Rigid Body Kinematics: Velocity of Arbitrary Point P

- In the Geometric Vector world:

\[ \mathbf{v}^P = \frac{d\mathbf{r}^P}{dt} = \mathbf{\dot{r}} + \mathbf{\dot{s}}^P = \mathbf{\dot{r}} + \mathbf{\omega} \times \mathbf{s}^P \]

- Using the Algebraic Vector representation (Chrono):

\[ \mathbf{\dot{r}}^P = \mathbf{\dot{r}} + \mathbf{\dot{s}}^P = \mathbf{\dot{r}} + \mathbf{\dot{A}}\mathbf{s}^P = \mathbf{\dot{r}} + \mathbf{\tilde{\omega}}\mathbf{A}\mathbf{s}^P = \mathbf{\dot{r}} + \mathbf{\tilde{\omega}}\mathbf{s}^P \]

- In plain words: the velocity \( \mathbf{\dot{r}}^P \) of a point P is equal to the sum of the velocity \( \mathbf{\dot{r}} \) of the point where the L-RF is located and the velocity \( \mathbf{\tilde{\omega}}\mathbf{s}^P \) due to the rotation with angular velocity \( \mathbf{\omega} \) of the rigid body.
3D Rigid Body Kinematics: Acceleration of Arbitrary Point P

- In the Geometric Vector world, by definition:

\[
\mathbf{a}^P \equiv \frac{d^2\mathbf{r}^P}{dt^2} = \ddot{\mathbf{r}} + \mathbf{\omega} \times \mathbf{\omega} \times \mathbf{s}^P + \mathbf{\omega} \times \mathbf{s}^P
\]

- Using the Algebraic Vector representation (Chrono):

\[
\mathbf{a}^P \equiv \ddot{\mathbf{r}}^P = \ddot{\mathbf{r}} + \ddot{\mathbf{s}}^P = \ddot{\mathbf{r}} + \mathbf{\omega}\mathbf{\omega} \mathbf{s}^P + \mathbf{\omega} \mathbf{\omega} \mathbf{s}^P = \ddot{\mathbf{r}} + \mathbf{\omega}\mathbf{\omega} \mathbf{s}^P + \mathbf{\omega} \mathbf{s}^P
\]
Putting Things in Perspective: What We’ve Covered so Far

• Discussed how to get the expression of a geometric vector in a “destination” reference frame knowing its expression in a “source” reference frame
  • Done via rotation matrix A

• Euler Parameters: a way of computing the A matrix when knowing the axis of rotation and angle of rotation

• Rate of change of the orientation matrix A → led to the concept of angular velocity

• Position, velocity and acceleration of a point P attached to a rigid body
Looking Ahead

• Kinematic constraints; i.e., joints

• Formulating the equations of motion
New Topic: Kinematic Constraints

• Geometric Constraint (GCon): a real world geometric attribute of the motion of the mechanical system
  • Examples:
    • Particle moves around point (1,2,3) on a sphere of radius 2.0
    • A unit vector \( \mathbf{u}_6 \) on body 6 is perpendicular on a certain unit vector \( \mathbf{u}_9 \) on body 9
    • The \( y \) coordinate of point Q on body 8 is 14.5

• Algebraic Constraint Equations (ACEs): in the virtual world, a collection of one or more algebraic constraints, involving the generalized coordinates of the mechanism and possibly time \( t \), that capture the geometry of the motion as induced by a certain Geometric Constraint
  • Examples:
    • \((x - 1)^2 + (y - 2)^2 + (z - 3)^2 - 4 = 0\)
    • \( \mathbf{u}_6^T \cdot \mathbf{u}_9 = 0 \)
    • \( [0 \ 1 \ 0] \cdot r_8^Q - 14.5 = 0 \)

• Modeling: the process that starts with the idealization of the real world to yield a GCon and continues with the GCon abstracting into a set of ACEs
Basic Geometric Constraints (GCons)

• We have four basic GCons:
  • DP1: the dot product of two vectors on two bodies is specified
  • DP2: the dot product of a vector of on a body and a vector between two bodies is specified
  • D: the distance between two points on two different bodies is specified
  • CD: the difference between the coordinates of two bodies is specified

• Note:
  • DP1 stands for Dot Product 1
  • DP2 stands for Dot Product 2
  • D stands for distance
  • CD stands for coordinate difference
Basic GCon: DP1

- Geometrically:
  \[ \bar{a}_i \cdot \bar{a}_j - f(t) = 0 \]

- Algebraically (matrix-vector notation):
  \[ \Phi^{DP1}(i, \bar{a}_i, j, \bar{a}_j, f(t)) = \bar{a}_i^T A_i^T A_j \bar{a}_j - f(t) = 0 \]
Basic GCon: DP2

- Geometrically:
  \[ \bar{a}_i \cdot \bar{d}_{ij} - f(t) = 0 \]

- Algebraically (matrix-vector notation):
  \[
  \Phi^{DP2}(i, \bar{a}_i, \bar{s}_i^P, j, \bar{s}_j^Q, f(t)) = \bar{a}_i^T A_i^T d_{ij} - f(t) \\
  = \bar{a}_i^T A_i^T (r_j + A_j \bar{s}_j^Q - r_i - A_i \bar{s}_i^P) - f(t) = 0
  \]
Basic GCon: D

- Geometrically:
  \[ \vec{d}_{ij} \cdot \vec{d}_{ij} - f^2(t) = 0 \]

- Algebraically (matrix-vector notation):
  \[
  \Phi^D(i, \vec{s}_i^P, j, \vec{s}_j^Q, f(t)) = d_{ij}^T d_{ij} - f^2(t) = 0
  \]
  \[
  = (r_j + A_j \vec{s}_j^Q - r_i - A_i \vec{s}_i^P)^T (r_j + A_j \vec{s}_j^Q - r_i - A_i \vec{s}_i^P) - f^2(t) = 0
  \]
Basic GCon: CD

- Geometrically (\( \mathbf{c} \) is a constant vector):
  \[
  \mathbf{c} \cdot (\mathbf{a}_j - \mathbf{a}_i) - f(t) = 0
  \]

- Algebraically (matrix-vector notation):
  \[
  \Phi^{\text{CD}}(\mathbf{c}, i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) = \mathbf{c}^T \mathbf{d}_{ij} - f(t) = \mathbf{c}^T (\mathbf{r}_j + \mathbf{A}_j \bar{\mathbf{s}}_j^Q - \mathbf{r}_i - \mathbf{A}_i \bar{\mathbf{s}}_i^P) - f(t) = 0
  \]
Intermediate GCons

• Two Intermediate GCons:
  • \( \perp_1 \): a vector is perpendicular on a plane belonging to a different body
  • \( \perp_2 \): a vector between two bodies is perpendicular to a plane belonging to the different body
Intermediate GCon: \( \perp 1 \) (Perpendicular Type 1)

- Geometrically, the motion is such that a vector \( \mathbf{c}_j \) on body \( j \) is perpendicular to a plane of body \( i \) that is defined by \( \mathbf{a}_i \) and \( \mathbf{b}_i \).

- Algebraically (matrix-vector notation):

\[
\Phi^{\perp 1}(i, \bar{a}_i, \bar{b}_i, j, \bar{c}_j) = \begin{bmatrix}
\Phi^{DP1}(i, \bar{a}_i, j, \bar{c}_j, 0) \\
\Phi^{DP1}(i, \bar{b}_i, j, \bar{c}_j, 0)
\end{bmatrix} = \begin{bmatrix}
\bar{a}_i^T A_i^T A_j \bar{c}_j \\
\bar{b}_i^T A_i^T A_j \bar{c}_j
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]
Intermediate GCon: \( \perp 2 \) (Perpendicular Type 2)

- Geometrically, a vector \( \overrightarrow{P_iQ_j} \) from body \( i \) to body \( j \) remains perpendicular to a plane defined by two vectors \( \vec{a}_i \) and \( \vec{b}_i \).

- Algebraically (matrix-vector notation):

\[
\Phi^{\perp 2}(i, \vec{a}_i, \vec{b}_i, \vec{s}_i^P, j, \vec{s}_j^Q) = \begin{bmatrix} \Phi^{DP2}(i, \vec{a}_i, \vec{s}_i^P, j, \vec{s}_j^Q, 0) \\ \Phi^{DP2}(i, \vec{b}_i, \vec{s}_i^P, j, \vec{s}_j^Q, 0) \end{bmatrix} = \begin{bmatrix} \vec{a}_i^T \mathbf{A}_i^T d_{ij} \\ \vec{b}_i^T \mathbf{A}_i^T d_{ij} \end{bmatrix} = 0
\]
High Level GCons

• High Level GCons also called joints:
  • Spherical Joint (SJ)
  • Universal Joint (UJ)
  • Cylindrical Joint (CJ)
  • Revolute Joint (RJ)
  • Translational Joint (TJ)
  • Other composite joints (spherical-spherical, translational-revolute, etc.)
High Level GCon: SJ [Spherical Joint]

\[
\Phi^{SJ} = \begin{bmatrix}
\Phi^{CD}(i, i, s^P_i, j, s^Q_j, 0) \\
\Phi^{CD}(j, i, s^P_i, j, s^Q_j, 0) \\
\Phi^{CD}(k, i, s^P_i, j, s^Q_j, 0)
\end{bmatrix}
\]
High Level GCon: CJ [Cylindrical Joint]

\[
\Phi_{CJ} = \begin{bmatrix}
\Phi_{1}^{-1}(i, a_i, \bar{b}_i, j, \bar{c}_j) \\
\Phi_{2}^{-1}(i, \bar{a}_i, \bar{b}_i, \bar{s}_j^P, j, \bar{s}_j^Q)
\end{bmatrix}
\]
High Level GCon: TJ [Translational Joint]
High Level GCon: RJ [Revolute Joint]
High Level GCon: UJ [Universal Joint]

\[
\Phi_{UJ} = \begin{bmatrix}
\Phi^{SJ}(i, s^P_{i}, j, s^Q_{j}) \\
\Phi^{DP1}(i, \bar{a}_i, j, \bar{a}_j, 0)
\end{bmatrix}
\]

Figure 9.4.15  Singular behavior of universal joint.
## Connection Between Basic and Intermediate/High Level GCons

<table>
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- Note that there are other GCons that are used, but they see less mileage
Constraints Supported in Chrono
New Topic:
Formulating the Equations of Motion

• Road map, full derivation of constrained equations of motion
  
  • Step 1: Introduce the types of force acting on one body present in a mechanical system
    • Distributed
    • Concentrated
  
  • Step 2: Express the virtual work produced by each of these forces acting on one body
  
  • Step 3: Evaluate the virtual work for the entire mechanical system
  
  • Step 4: Apply principle of virtual work (via D’Alembert’s principle) to obtain the EOM
Generic Forces/Torques Acting on a Mechanical System

• Distributed forces
  • Inertia forces
  • Volume/Mass distributed force (like gravity, electromagnetic, etc.)
  • Internal forces

• Concentrated forces/torques
  • Reaction forces/torques (induces by the presence of kinematic constraints)
  • Externally applied forces and torques (me pushing a cart)
Virtual Work for One Body, Side Trip

- Quick example below only shows virtual work produced by the **inertial force**
  - Same recipe applied for all other forces, distributed or concentrated

- Starting point: consider point $P$ of body $i$ associated with infinitesimal mass element $dm_i(P)$

- Expression of the force:
  \[-\ddot{r}_i^P \, dm_i(P)\]

- Virtual work produced:
  \[[\delta r_i^P]^T \cdot [-\ddot{r}_i^P \, dm_i(P)]\]

- Body virtual work obtained by summing over all points $P$ of body $i$:
  \[
  \delta W = \int_{m_i} [-[\delta r_i^P]^T \cdot \ddot{r}_i^P \, dm_i(P)]
  \]

- Upon expressing virtual displacement of $P$ and its acceleration $\ddot{r}_i^P$:
  \[
  \delta W = \int_{m_i} [\delta r_i^T + \delta \pi_i^T \pi_i^P A_i^T] \cdot \left[ \ddot{r}_i + A_i \ddot{\bar{\omega}}_i \bar{\omega}_i \pi_i^P + A_i \ddot{\bar{\bar{\omega}}}_i \bar{\bar{\omega}}_i \pi_i^P \right] \, dm_i(P) = \delta r_i^T m_i \ddot{r}_i + \delta \pi_i^T \left[ \ddot{\bar{\omega}}_i \ddot{J}_i \bar{\omega}_i + \ddot{\bar{\bar{\omega}}}_i \ddot{J}_i \bar{\bar{\omega}}_i \right]
  \]
Final Form, Expression of Virtual Work

- When all said and done, the expression of the virtual work assumes the form:

\[
\delta W = \sum_{i=1}^{nb} \left[ -\delta r_i^T m_i \ddot{\mathbf{r}}_i - \delta \pi_i^T \ddot{\mathbf{\omega}}_i \mathbf{\bar{J}}_i \dot{\mathbf{\omega}}_i - \delta \pi_i^T \mathbf{\bar{J}}_i \dot{\mathbf{\omega}}_i + \delta r_i^T \cdot \mathbf{F}^m_i + \delta \pi_i^T \cdot \mathbf{n}^m_i \\
+ \delta r_i^T \mathbf{F}^a_i + \delta \pi_i^T \mathbf{n}^a_i + \delta r_i^T \mathbf{F}^r_i + \delta \pi_i^T \mathbf{n}^r_i \right] = 0
\]

- Alternatively,

\[
\delta W = \sum_{i=1}^{nb} \left[ \delta r_i^T (-m_i \ddot{\mathbf{r}}_i + \mathbf{F}^m_i + \mathbf{F}^a_i + \mathbf{F}^r_i) + \delta \pi_i^T (-\ddot{\mathbf{\omega}}_i \mathbf{\bar{J}}_i \dot{\mathbf{\omega}}_i - \mathbf{\bar{J}}_i \dot{\mathbf{\omega}}_i + \mathbf{n}^m_i + \mathbf{n}^a_i + \mathbf{n}^r_i) \right] = 0
\]
Moving from One Body to a Mechanical System

- Total virtual work, for the entire system, assumes the form:

\[
\delta W = \sum_{i=1}^{nb} \left[ -\delta \mathbf{r}_i^T \dot{\mathbf{r}}_i m_i - \delta \mathbf{\tilde{\pi}}_i^T \dot{\mathbf{\tilde{\omega}}}_i \mathbf{J}_i \dot{\mathbf{\tilde{\omega}}}_i - \delta \mathbf{\tilde{\pi}}_i^T \mathbf{\bar{J}}_i \dot{\mathbf{\tilde{\omega}}}_i + \delta \mathbf{r}_i^T \cdot \mathbf{F}_i^m + \delta \mathbf{\tilde{\pi}}_i^T \cdot \mathbf{\bar{n}}_i^m \\
+ \delta \mathbf{r}_i^T \mathbf{F}_i^a + \delta \mathbf{\tilde{\pi}}_i^T \mathbf{\bar{n}}_i^a + \delta \mathbf{r}_i^T \mathbf{F}_i^r + \delta \mathbf{\tilde{\pi}}_i^T \mathbf{\bar{n}}_i^r \right] = 0
\]

- Alternatively,

\[
\delta W = \sum_{i=1}^{nb} \left[ \delta \mathbf{r}_i^T \left( -\dot{\mathbf{r}}_i m_i + \mathbf{F}_i^m + \mathbf{F}_i^a + \mathbf{F}_i^r \right) + \delta \mathbf{\tilde{\pi}}_i^T \left( -\dot{\mathbf{\tilde{\omega}}}_i \mathbf{J}_i \dot{\mathbf{\tilde{\omega}}}_i - \mathbf{\bar{J}}_i \dot{\mathbf{\tilde{\omega}}}_i + \mathbf{\bar{n}}_i^m + \mathbf{\bar{n}}_i^a + \mathbf{\bar{n}}_i^r \right) \right] = 0
\]

- Recall that for each body \( i \), virtual translations \( \delta \mathbf{r}_i \) and virtual rotations \( \delta \mathbf{\tilde{\pi}}_i \) are arbitrary
Equations of Motion (EOM) for A System of Rigid Bodies

- Since equation on previous slide should hold for any set of virtual displacements \((\delta r_1, \delta \pi_1), (\delta r_2, \delta \pi_2), \ldots, (\delta r_{nb}, \delta \pi_{nb})\), then we necessarily have that for \(i = 1, \ldots, nb:\)

\[
-m_i \ddot{r}_i + F^m_i + F^a_i + F^r_i = 0_3
\]

\[
-\ddot{\omega}_i \vec{J}_i \ddot{\omega}_i - \vec{J}_i \dot{\omega}_i + \ddot{\pi}^m_i + \ddot{\pi}^a_i + \ddot{\pi}^r_i = 0_3
\]

- Equivalently, for \(i = 1, \ldots, nb\)

\[
m_i \ddot{r}_i = F^m_i + F^a_i + F^r_i
\]

\[
\vec{J}_i \dot{\omega}_i = \ddot{\pi}^m_i + \ddot{\pi}^a_i + \ddot{\pi}^r_i - \ddot{\omega}_i \vec{J}_i \ddot{\omega}_i
\]

- The set of equations above represent the EOM for the system of \(nb\) rigid bodies.
The Joints (Kinematic Constraints) Lead to Reaction Forces

- The collection of all \( nc \) kinematic and driving constraints – stack them together:

\[
\Phi(q, t) = \begin{bmatrix} \Phi^K(q) \\ \Phi^D(q, t) \end{bmatrix} = 0_{nc}
\]

- Recall that any one of the constraints in \( \Phi \) is one of the four basic GCons introduced earlier

- The variation of \( \Phi \): stack together the variation of each of the GCons that enters in \( \Phi \)

- A virtual displacement of the bodies in the system will lead to a virtual variation \( \delta \Phi \) that depends on the position and orientation of the bodies:

\[
\delta \Phi = \Phi_r \delta r + \Pi(\Phi) \delta p = 0_{nc}
\]

- In matrix form, we can express the above relations as

\[
\delta \Phi(r, p) = \begin{bmatrix} \Phi_r & \Pi(\Phi) \end{bmatrix} \cdot \begin{bmatrix} \delta r \\ \delta p \end{bmatrix} = \bar{R}(\Phi) \cdot \begin{bmatrix} \delta r \\ \delta p \end{bmatrix} = 0_{nc}
\]

- \( \Phi_r \) and \( \Pi(\Phi) \): the key ingredients needed to express the reaction forces induced by the constraints \( \Phi(q, t) = 0_{nc} \)
Switching to Matrix-Vector Notation

- Notation used to simplify expression of EOM:
  - $I_3$ is the identity matrix of dimension 3
  - $F_i^a$ and $F_i^m$ – applied and mass-distributed force, body $i$
  - $\mathbf{n}_i^a$ and $\mathbf{n}_i^m$ – applied and mass-distributed torque, body $i$
  - $m_i$ and $\bar{J}_i$ – mass and mass moment of inertia, body $i$

- Matrix-vector notation:

$$M = \begin{bmatrix} m_1 I_3 & 0_{3 \times 3} & \cdots & 0_{3 \times 3} \\ 0_{3 \times 3} & m_2 I_3 & \cdots & 0_{3 \times 3} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{3 \times 3} & 0_{3 \times 3} & \cdots & m_{nb} I_3 \end{bmatrix} \quad \bar{J} = \begin{bmatrix} \bar{J}_1 & 0_{3 \times 3} & \cdots & 0_{3 \times 3} \\ 0_{3 \times 3} & \bar{J}_2 & \cdots & 0_{3 \times 3} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{3 \times 3} & 0_{3 \times 3} & \cdots & \bar{J}_{nb} \end{bmatrix}$$

$$\mathbf{r} = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_{nb} \end{bmatrix}_{3nb} \quad \dot{\omega} = \begin{bmatrix} \mathbf{\dot{\omega}}_1 \\ \vdots \\ \mathbf{\dot{\omega}}_{nb} \end{bmatrix}_{3nb} \quad \mathbf{F} = \begin{bmatrix} F_i^a + F_i^m \\ \vdots \\ F_{nb}^a + F_{nb}^m \end{bmatrix}_{3nb} \quad \tau = \begin{bmatrix} \mathbf{n}_1^a + \mathbf{n}_1^m - \mathbf{\dot{\omega}}_1 \bar{J}_1 \mathbf{\dot{\omega}}_1 \\ \vdots \\ \mathbf{n}_{nb}^a + \mathbf{n}_{nb}^m - \mathbf{\dot{\omega}}_{nb} \bar{J}_{nb} \mathbf{\dot{\omega}}_{nb} \end{bmatrix}_{3nb}$$
EOM: the Newton-Euler Form

- According to Lagrange Multiplier theorem, there exists a vector of Lagrange Multipliers, \( \lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_{nc} \end{bmatrix} \), so that

\[
\begin{bmatrix}
\mathbf{M}\ddot{\mathbf{r}} - \mathbf{F} \\
\mathbf{J}\ddot{\omega} - \tau
\end{bmatrix} + \begin{bmatrix}
\Phi_\tau^T \\
\mathbf{P}^T(\Phi)
\end{bmatrix} \lambda = \mathbf{0}_{6nb}
\]

- Expression above: **Newton-Euler form of the EOM.** Equivalently expressed as:

\[
\left\{
\begin{array}{c}
\mathbf{M}\ddot{\mathbf{r}} + \Phi_\tau^T \lambda = \mathbf{F} \\
\mathbf{J}\ddot{\omega} + \mathbf{P}^T(\Phi)\lambda = \tau
\end{array}
\right.
\]